General Dynamic Wormholes and Violation of the Null Energy Condition

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Abstract. Although wormholes can be treated as topological objects in spacetime and from a global point-of-view, a precise definition of what a wormhole throat is and where it is located can be developed and treated entirely in terms of local geometry. This has the advantage of being free from unnecesary technical assumptions about asymptotic flatness, and other global properties of the spacetime containing the wormhole. We discuss our recent work proving that the violation of the null energy condition (NEC) is a *generic* feature of all wormholes, whether they be time-dependent or static, and demonstrate that time-dependent wormholes have *two* throats, one for each direction through the wormhole, which coalesce only in the static limit.

1 Introduction

The fact that traversable wormholes are accompanied by unavoidable violations of the null energy condition (NEC) is perhaps one of the most important aspects of Lorentzian wormhole physics [1,2,3]. The original proof of the necessity for NEC violations at or near the throat of a traversable wormhole was limited to the static spherically symmetric Morris-Thorne wormhole [1], though it was soon after realized that NEC violations typically occurred in at least some explicit examples of static non-symmetric [4] and spherically-symmetric time-dependent [5] wormholes. A considerably more general proof of the necessity of NEC violations was provided by the *topological censorship theorem* of Friedman, Schleich, and Witt [6] though this theorem requires many technical assumptions concerning asymptotic flatness and causality conditions that limit its applicability.

We have recently adopted a different strategy by developing new general theorems concerning energy condition violations at and near the throat of traversable wormholes [7,8], by focusing attention only on the local behavior of the geometry at and near the throat, and dispensing with all assumptions about symmetry, asymptotic behaviour, and causal properties. This strategy was inspired by the fact that there are many classes of spacetime configurations that we would meaningfully wish to call wormholes that possess either trivial topology [3,9] or do not necessarily possess asymptotically flat

regions [10]. The strategy that we have developed views a wormhole throat as a marginally anti-trapped surface [11,12] that is, a closed two-dimensional spatial hypersurface such that one of the two future-directed null geodesic congruences orthogonal to it is just beginning to diverge. Physically, this definition relies only on the properties of bundles of light rays passing through the region surrounding the throat, and not on how that throat may or may not embed in some fictitious Euclidean embedding space.

2 Definition of generic wormhole throats

For the generic but *static* case, the throat was defined as a two-dimensional hypersurface of minimal area [7,8]. The time independence allows one to locate that minimal hypersurface entirely within one of the constant-time three-dimensional spatial slices, and the conditions of extremality and minimality can be applied and enforced within that single time-slice. For a static throat, variational principles involve performing arbitrary time-independent surface deformations of the hypersurface in the remaining spatial direction orthogonal to the hypersurface, which can always be taken to be locally Gaussian. By contrast, in the time-dependent case, it may not be possible to define the throat by working within one time slice: the dynamic throat is an extended object in spacetime, and the variational principle must be carried out employing surface deformations in the two independent *null* directions orthogonal to the hypersurface: say, δu_+ and δu_-. This, by the way, demonstrates why it is that *in general* the embedding of the spatial part of a wormhole spacetime in an Euclidean $\mathbf{R}^n$ is no longer a reliable operational technique for defining "flare-out" in the time-dependent case. We will come back to this point below. Of course, in the static limit these two variations will no longer be independent and arbitrary deformations in the two null directions reduce to a single variation in the constant-time spatial direction as demonstrated below.

2.1 Geometric Preliminaries

We now set up and define the properties of throats in terms of the null congruences. Bear in mind that a throat will be characterized in terms of the behavior of a single set of null geodesics orthogonal to it. We define a wormhole throat Σ_{u+} (there is also one for the other null congruence) to be a closed 2-dimensional hypersurface of minimal area taken in one of the constant-u_+ slices, where u_+ is an affine parameter suitable for parameterizing the future-directed null geodesics l_+ orthogonal to Σ_{u+}. All this means is that we imagine "starting" off a collection of light pulses along the hypersurface and we can always arrange the affine parameterizations of each pulse to be equal to some constant on the hypersurface; we take this constant to be zero. We wish to emphasize that there is a corresponding definition for

the other throat Σ_{u-}. In the following, we define and develop the conditions that both hypersurfaces must satisfy individually to be considered as throats, and shall do so in a unified way by treating them together by employing the $\pm$-label. Our next task is to compute the hypersurface areas and impose the conditions of extremality and minimality directly and to express these constraints in terms of the expansion θ of the null geodesics. The area of the two-dimensional spatial hypersurface $\Sigma_{u\pm}$ is given by

$$A(\Sigma_{u\pm}) = \int_{\Sigma_{u\pm}} \sqrt{\gamma}\, d^2x. \tag{1}$$

An arbitrary variation of the surface with respect to deformations in the null direction parameterized by $u_\pm$ is

$$\delta A(\Sigma_{u\pm}) = \int_{\Sigma_{u\pm}} \frac{d\sqrt{\gamma}}{du_\pm}\, \delta u_\pm(x)\, d^2x.$$
$$= \int_{\Sigma_{u\pm}} \sqrt{\gamma}\, \frac{1}{2} \gamma^{ab} \frac{d\gamma_{ab}}{du_\pm}\, \delta u_\pm(x)\, d^2x. \tag{2}$$

If this is to vanish for arbitrary variations $\delta u_\pm(x)$, then we must have that

$$\frac{1}{2} \gamma^{ab} \frac{d\gamma_{ab}}{du_\pm} = 0, \tag{3}$$

which expresses the fact that the hypersurface $\Sigma_{u\pm}$ is extremal.

This condition of hypersurface extremality can also be phrased equivalently and directly in terms of the expansion of the null congruences. The simplest way to do so is to consider the Lie derivative $\mathcal{L}_l^\pm$ acting on the full spacetime metric:

$$\begin{aligned}\mathcal{L}_l^\pm g_{ab} &= l_\pm^c \nabla_c g_{ab} + g_{cb} \nabla_a l_\pm^c + g_{ac} \nabla_b l_\pm^c \\ &= \nabla_a l_{\pm b} + \nabla_b l_{\pm a} \\ &= B^\pm{}_{ba} + B^\pm{}_{ab} = 2B^\pm{}_{(ab)},\end{aligned} \tag{4}$$

with the second equality holds due to the covariant constancy of the metric. The third line defines the tensor field B_{ab} as the covariant derivative of the future-directed null vectors (there is one such tensor field for each null congruence):

$$B_{ab}^\pm \equiv \nabla_b l_{\pm a}, \tag{5}$$

We now use the decomposition (7) of the spacetime metric following the description of Carter [13]. The future-directed "outgoing" null vector l_+^a and future-directed "ingoing" null vector l_-^a introduced above together with a spatial orthogonal projection tensor γ^{ab} can be chosen satisfying the following relations:

$$\begin{aligned} l_+^a l_{+a} &= l_-^a l_{-a} = 0, \qquad l_+^a l_{-a} = l_-^a l_{+a} = -1 \\ l_\pm^a \gamma_{ab} &= 0, \qquad \gamma_c^a \gamma^{cd} = \gamma^{ad}. \end{aligned} \tag{6}$$

In terms of these null vectors and projector, we can decompose the full space-time metric (indeed, any tensor) uniquely:

$$g_{ab} = \gamma_{ab} - l_{-a}l_{+b} - l_{+a}l_{-b}. \tag{7}$$

Physically, this decomposition leads to a parameterization of spacetime points in terms of two spatial coordinates (typically denoted x) plus two null coordinates [$u_\pm$, or sometimes (u, v)]. (We do not want to prejudice matters by taking the words "outgoing" and "ingoing" too literally, since outside and inside do not necessarily make much sense in situations of nontrivial topology. The critical issue is that the spacelike hypersurface must have two sides and $+$ and $-$ are just two convenient labels for the two null directions.)

Using (4) we can now work out the Lie derivative using the Leibnitz rule:

$$\begin{aligned}
B^{\pm}{}_{(ab)} &= \frac{1}{2}\mathcal{L}_l^{\pm}g_{ab} \\
&= \frac{1}{2}\mathcal{L}_l^{\pm}(\gamma_{ab} - l_{-a}l_{+b} - l_{+a}l_{-b}), \\
&= \frac{1}{2}\mathcal{L}_l^{\pm}\gamma_{ab} - \frac{1}{2}(l_{-a}\mathcal{L}_l^{\pm}l_{+b} + l_{+b}\mathcal{L}_l^{\pm}l_{-a} + (a \leftrightarrow b)),
\end{aligned} \tag{8}$$

from which, and using the properties in (6), implies

$$\begin{aligned}
\theta_\pm &\equiv g^{ab}B^{\pm}{}_{(ab)} = \gamma^{ab}B^{\pm}{}_{(ab)} \\
&= \frac{1}{2}\gamma^{ab}\mathcal{L}_l^{\pm}\gamma_{ab} = \frac{1}{2}\gamma^{ab}\frac{d\gamma_{ab}}{du_\pm}.
\end{aligned} \tag{9}$$

Note the trace of the symmetrized tensor $B_{(ab)}$ defines the divergence θ. So the condition that the area of the hypersurface be extremal (3) is simply that the expansion of the null geodesics vanish at the surface: $\theta_\pm = 0$.

2.2   Flare-out

To ensure that the area be *minimal*, we need to impose an additional constraint and shall require that $\delta^2 A(\Sigma_{u\pm}) \geq 0$. By explicit computation,

$$\begin{aligned}
\delta^2 A(\Sigma_{u\pm}) &= \int_{\Sigma_{u\pm}} \sqrt{\gamma}\left(\theta_\pm{}^2 + \frac{d\theta_\pm}{du_\pm}\right)\delta u_\pm(x)\,\delta u_\pm(x)d^2x \\
&= \int_{\Sigma_{u\pm}} \sqrt{\gamma}\frac{d\theta_\pm}{du_\pm}\,\delta u_\pm(x)\,\delta u_\pm(x)\,d^2x \geq 0,
\end{aligned} \tag{10}$$

where we have used the extremality condition ($\theta_\pm = 0$) in arriving at this last inequality. For this to hold at the throat for arbitrary variations $\delta u_\pm(x)$, it follows from $(\delta u_\pm(x))^2 \geq 0$, that we must have

$$\frac{d\theta_\pm}{du_\pm} \geq 0, \tag{11}$$

in other words, the expansion of the cross-sectional area of the future-directed null geodesics must be locally increasing at the throat. This is the precise generalization of the Morris-Thorne "flare-out" condition to arbitrary wormhole throats. This makes eminent good sense since the expansion is the measure of the cross-sectional area of bundles of null geodesics, and a positive derivative indicates that this area is locally increasing or "flaring-out" as one moves along the null direction. Note that this definition is free from notions of embedding and "shape"-functions as well as global features of the spacetime. So in general, we have to deal with two throats: Σ_{u+} such that $\theta_+ = 0$ and $d\theta_+/du_+ \geq 0$ and Σ_{u-} such that $\theta_- = 0$ and $d\theta_-/du_- \geq 0$. We shall soon see that for static wormholes the two throats coalesce and this definition automatically reduces to the static case considered in [7,8]. The logical development reviewed here closely parallels that of the static case though there are important differences.

The conditions that a wormhole throat be both extremal and minimal are the simplest requirements that one would want a putative throat to satisfy and which may be summarized in the following definition (in the following, the hypersurfaces are understood to be closed and spatial). Since these definitions hold of course for both throats, we momentarily drop the distinction and suppress the $\pm$ label.

Definition: Simple flare-out condition *A two-surface satisfies the "simple flare-out" condition if and only if it is extremal,* $\theta = 0$, *and also satisfies* $d\theta/du \geq 0$. The characterization of a generic wormhole throat in terms of the expansion of the null geodesics shows that any two-surface satisfying the simple flare-out condition is a *marginally anti-trapped surface*, where the notion of trapped surfaces is a familiar concept that arises primarily in the context of singularity theorems, gravitational collapse and black hole physics [14,15].

Generically, we would expect the inequality $\delta^2 A(\Sigma_u) > 0$ to be strict, so that the surface is truly a minimal (not just extremal) surface. This will pertain provided the inequality $d\theta/du > 0$ is a strict one for at least *some* points on the throat. This suggests the following definition.

Definition: Strong flare-out condition *A two-surface satisfies the "strong flare-out" condition at the point x if and only if it is extremal,* $\theta = 0$, *satisfies* $\frac{d\theta}{du} \geq 0$ *everywhere on the surface and if at the point x, the inequality is strict:*

$$\frac{d\theta}{du} > 0. \tag{12}$$

If the latter strict inequality holds for all $x \in \Sigma_u$ in the surface, then the wormhole throat is seen to correspond to a *strongly anti-trapped surface.* It is sometimes sufficient and convenient to work with a weaker, integrated forms of the flare-out condition. These are described in detail in [12].

2.3   Static limit

In a static spacetime, a wormhole throat is a closed two-dimensional spatial hypersurface of minimal area that, without loss of generality, can be located entirely within a single constant-time spatial slice [7,8]. Now, for any static spacetime, one can always decompose the spacetime metric in a block-diagonal form as

$$g_{ab} = -V_a V_b + {}^{(3)}g_{ab}, \qquad (13)$$

where $V^a = \exp[\phi](\frac{\partial}{\partial t})^a$ is a timelike vector field orthogonal to the constant-time spatial slices and $\phi = \phi(x)$ is some function of the spatial coordinates only. In the vicinity of the throat we can always set up a system of Gaussian coordinates n so that

$$ {}^{(3)}g_{ab} = n_a n_b + \gamma_{ab}, \qquad (14)$$

where $n^a = (\frac{\partial}{\partial n})^a$, $n^a n_a = +1$, and γ_{ab} is the two-metric of the hypersurface. Putting these facts together implies that in the vicinity of any static throat we may write the spacetime metric as

$$g_{ab} = -V_a V_b + n_a n_b + \gamma_{ab}. \qquad (15)$$

But (7) holds in general, so comparing both metric representations yields the identity

$$-l^a_- l^b_+ - l^a_+ l^b_- = V^a V^b + n^a n^b, \qquad (16)$$

and the following (linear) transformation relates the two metric decompositions and preserves the inner-product relations in (6):

$$l^a_- = \frac{1}{2}(V^a + n^a), \quad l^a_+ = \frac{1}{2}(V^a - n^a). \qquad (17)$$

Since the throat is static, γ_{ab} is time-independent, hence when we come to vary the area (1) with respect to arbitrary perturbations in the two independent null directions we find that

$$\frac{\partial \gamma_{ab}}{\partial u_+} \delta u_+ = \frac{1}{2}\left(\exp[\phi]\frac{\partial \gamma_{ab}}{\partial t}\delta t + \frac{\partial \gamma_{ab}}{\partial n}\delta n\right) = \frac{1}{2}\frac{\partial \gamma_{ab}}{\partial n}\delta n,$$

$$\frac{\partial \gamma_{ab}}{\partial u_-} \delta u_- = \frac{1}{2}\left(\exp[\phi]\frac{\partial \gamma_{ab}}{\partial t}\delta t - \frac{\partial \gamma_{ab}}{\partial n}\delta n\right) = -\frac{1}{2}\frac{\partial \gamma_{ab}}{\partial n}\delta n. \qquad (18)$$

Thus the two variations in the null directions are no longer independent, and reduce to taking a single surface variation in the spatial Gaussian direction. So, $\theta_+ = 0 \Longleftrightarrow \theta_- = 0$ at the same hypersurface, proving that $\Sigma_{u_+} = \Sigma_{u_-}$ in the static limit, and so static wormholes have only one throat. A thorough analysis of the geometric structure of the generic static traversable wormhole can be found in [7,8].

3   Constraints on the stress-energy

With the definition of wormhole throat made precise we now turn to derive constraints that the stress energy tensor must obey on (or near) any wormhole throat. The constraints follow from combining the Raychaudhuri equation [13,14,15] governing the rate-of-change of the divergence along the null direction (there is one for the (+)-congruence and one for the (-)-congruence)

$$\frac{d\theta_{\pm}}{du_{\pm}} = -\frac{1}{2}\theta_{\pm}{}^2 - \sigma^{\pm ab}\sigma_{\pm ab} + \omega^{\pm ab}\omega_{\pm ab} - R^d_c l^c_{\pm} l_{\pm d}, \tag{19}$$

with the flare-out conditions (11) or (12). We can use the Einstein equation $(R_{ab} - \frac{1}{2}g_{ab}R = 8\pi T_{ab})$ to cast this into an equation involving the stress-energy. Here, σ_{ab} and ω_{ab} denote the symmetric shear and antisymmetric twist of the null congruence. They are purely spatial tensors. It is clear that these constraints apply with equal validity at both the + and − throats, and in the following we cover both classes simultaneously and without risk of confusion by dropping the ±-labels.

Since all throats are extremal hypersurfaces ($\theta = 0$) the Raychaudhuri equation (19) evaluated at the throat reduces to

$$\frac{d\theta}{du} + \sigma^{ab}\sigma_{ab} = -8\pi T_{ab}\, l^a l^b, \tag{20}$$

where we have used the Einstein equation and the fact that the null geodesic congruences are hypersurface orthogonal, so that the twist $\omega_{ab} = 0$ vanishes identically on the throat. We make no claim regarding the shear, nor do we need to, except to point out that since σ_{ab} is purely spatial, its square $\sigma^{ab}\sigma_{ab} \geq 0$ is positive semi-definite everywhere (not just on the throat). Consider a marginally anti-trapped surface, *i.e.*, a throat satisfying the simple flare-out condition. Then the stress energy tensor on the throat must satisfy

$$T_{ab}\, l^a l^b \leq 0. \tag{21}$$

The NEC is therefore either violated, or on the verge of being violated $(T_{ab}\, l^a l^b \equiv 0)$, on the throat. Of course, whichever one of the two null geodesic congruences (l_+ or l_-) you are using to define the wormhole throat (anti-trapped surface), you must use the *same* null geodesic congruence for deducing null energy condition violations.

For throats satisfying the strong flare-out condition, we have instead the stronger statement that for all points on the throat,

$$T_{ab}l^a l^b \leq 0, \text{ and } \exists x \in \Sigma_u \text{ such that } T_{ab}\, l^a l^b < 0, \tag{22}$$

so that the NEC is indeed violated for at least *some* points lying on the throat. By continuity, if $T_{ab}\, l^a l^b < 0$ at x, then it is strictly negative within

a finite open neighborhood of x: $B_\epsilon(x)$. Finally, for throats that are strongly anti-trapped surfaces, we derive the most stringent constraint stating that

$$T_{ab}\, l^a l^b < 0 \ \ \forall x \in \Sigma_u, \tag{23}$$

so that the NEC is violated *everywhere* on the throat.

What can we say about the energy conditions in the region surrounding the throat? This requires knowledge of the expansion, shear and twist in the neighborhood of the throat. Luckily, we can dispense with the twist immediately. Indeed, the twist equation [14] is a simple, first-order linear differential equation:

$$\frac{d\omega_{ba}}{du} = -\theta\omega_{ba} - 2\sigma^c_{[a}\omega_{b]c}, \tag{24}$$

whose exact solution (if somewhat formal in appearance) is

$$\omega_{ab}(u) = \exp\left(-\int_0^u \theta(s)ds\right)\,\mathcal{U}_a{}^c(u)\,\mathcal{U}_b{}^d(u)\,\omega_{cd}(0), \tag{25}$$

where the quantity $\mathcal{U}(u)$ denotes the path-ordered exponential

$$\mathcal{U}_a{}^c(u) = \mathcal{P}\exp\left(-\int_0^u \sigma\, ds\right)_a{}^c. \tag{26}$$

So, an initially hypersurface orthogonal congruence remains twist-free everywhere, both on and off the throat: $\omega_{ba}(0) = 0 \Rightarrow \omega_{ba}(u) = 0$. Then the equation $\frac{d\theta}{du} + \frac{1}{2}\theta^2 + \sigma^{ab}\sigma_{ab} = -8\pi T_{ab}\, l^a l^b$, is seen to be valid for all u. Coming back to simply-flared throats, we have two pieces of information regarding the expansion: namely that $\theta(0) = 0$ and $(d\theta(u)/du)_{u=0} \geq 0$, so that if we expand θ in a neighborhood of the throat then we have that $\frac{d\theta(u)}{du} = \left.\frac{d\theta(u)}{du}\right|_{u=0} + O(u)$, so over each point x on the throat, there exists a finite range in affine parameter $u \in (0, u_x^*)$ for which $\frac{d\theta(u)}{du} \geq 0$. Since both θ^2 and $\sigma^{ab}\sigma_{ab}$ are positive semi-definite, we conclude that the stress-energy is either violating, or on the verge of violating, the NEC along the partial null curve $\{x\} \times (0, u_x^*)$ based at x. If the throat is of the strongly-flared variety, then we see that the NEC is definitely violated at least over some finite regions surrounding the throat: $\bigcup_x \{x\} \times (0, u_x^*)$, and including the base points x. For strongly anti-trapped surfaces, the NEC is violated everywhere in a finite region surrounding the entire throat, and including the throat itself.

4   Discussion

The familiar flare-out property characterizing wormholes is manifested in the properties of light rays (null geodesics) that traverse a wormhole: bundles of light rays that enter the wormhole at one mouth and exit from the other must have cross-sectional area that first decreases, reaching a true minimum

at the throat, and then increases. These properties can be quantified precisely in terms of the expansion $\theta_{\pm}$ of the (future-directed) null geodesics together with its derivative $d\theta_{\pm}/du_{\pm}$, where all quantities are evaluated at the two-dimensional spatial hypersurface comprising the throat. Strictly speaking, this flaring-out behavior of the outgoing null geodesics (l_{+}) defines one throat: the "outgoing" throat. But one can also ask for the flaring-out property to be manifested in the propagation of the set of ingoing null geodesics (l_{-}) as they traverse the wormhole, and this leads one to define a second, or "ingoing" throat. In general, these two throats need not be identical (which can give rise to interesting causal properties [11]), but for the static limit they do coalesce and are indistinguishable.

The flaring-out property implies that all wormhole throats are in fact *anti-trapped* surfaces, an identification that was anticipated some time ago by Page [16]. With this definition and using the Raychaudhuri equation, we are able to place rigorous constraints on the Ricci tensor and the stress-energy tensor at the throat(s) of the wormhole as well as in the regions near the throat(s). We find, as expected, that wormhole throats generically violate the null energy condition and we have provided rigorous results regarding this matter. This should now settle the issue of energy condition violations for wormholes.

Until recently, the nature of the energy-condition violations associated with wormhole throats has led numerous authors to try to find ways of evading or minimizing the violations. Most attempts to do so focus on alternative gravity theories in which one may be able to force the extra degrees of freedom to absorb the energy-condition violations (some of these scenarios are discussed in [8], see also [17,18]). But the energy condition violations are still always present, for sweeping the energy condition violations into a particular sector does not make the "problem" go away. More recently it has been realized that time-dependence lets one move the energy condition violating regions around in *time* [19,20,21,22,23]. However, temporary suspension of the violation of the NEC at a time-dependent throat also leads to a simultaneous obliteration of the flare-out property of the throat itself [12], so this strategy ends up destroying the throat and nothing is to be gained. (See also [8].) It is crucial to note that we have defined flare-out in terms of the expansion properties of light rays at the throat and *not* in terms of so-called "shape" functions or embedding diagrams. While the latter can certainly be used without risk[1] for detecting flare-out in static wormholes, they are at

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[1] But even the static case requires that due care be exercised. By the Whitney embedding theorem [24], a subset $X \subset \mathbf{R}^n$ embeds in an $\mathbf{R}^{2n}$. So we should expect a static throat, which is two dimensional, to embed in an $\mathbf{R}^4$. The fact that embeddings of most of the static wormholes studied so far can be carried out in an $\mathbf{R}^3$ is due to the highly symmetric nature of the wormholes chosen for study. A counterexample is provided by the Klein bottle, which can be *visualized* but not embedded in $\mathbf{R}^3$, where it self-intersects.

best misleading if applied to dynamic wormholes. This is simply because the embedding of a wormhole spacetime requires selecting and lifting out a particular time-slice and embedding this instantaneous spatial three-geometry in a flat Euclidean $\mathbf{R}^n$. For a static wormhole, any constant time-slice will suffice, and if the embedded surface is flared-out in the spatial direction orthogonal to the throat, then it is flared-out in spacetime as well. But if the wormhole is dynamic, flare-out in the spatial direction does not imply flare-out in the *null* directions orthogonal to the throat.

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